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Translated by J. J. D.

NONLINEAR ACOUSTICS OF CHEMICALLY ACTIVE MEDIA

PMM Vol. 35, №6, 1971, pp.1023-1037

O. S. RYZHOV

(Moscow)

(Received June 16, 1971)

Nonlinear propagation of disturbances is examined in reacting mixtures where the change of composition is determined by the course of a single chemical reaction. Depending on the relationship between macroscopic time and relaxation time, we distinguish two basic types of processes: quasi-frozen and quasi-equilibrium. Media are examined also, in which the frozen and equilibrium speeds of sound are nearly equal in magnitude. Solutions are constructed for asymptotic equations which describe the flow parameters behind shock fronts and in expansion waves. A mathematical analogy is formulated for the effect of rates of chemical reactions, the effect of "longitudinal viscosity", and the effect of thermal conductivity on the structure of the perturbed field.

1. Initial equations. It will be assumed that in the flow of chemically active gas mixture only one reaction takes place. The change in the composition of the mixture is then characterized by a single parameter q which is called completeness of reaction.

The equations of motion of the mixture are taken in the form [1]

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + (v - 1) \frac{\rho v}{r} = 0, \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \\ Q \left(\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial r} \right) + T \left(\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial r} \right) = 0, \quad \frac{\partial q}{\partial t} + v \frac{\partial q}{\partial r} = q^* \end{aligned} \quad (1.1)$$

Here t is the time, r is the distance from the plane, axis or center of symmetry, v is the velocity, ρ is the density, p is the pressure, s is the specific entropy, T is the temperature, q and Q are the rate and affinity of chemical reaction. The parameter $v = 1, 2, 3$ for flows with a plane, axis or center of symmetry, respectively.

In order to close the system it is necessary to introduce three additional equations which connect thermodynamic functions q, ρ, p, s, Q and e . According to the Gibbs relationship the increase in specific internal energy e is

$$de = Qdq - pdV + Tds, \quad V = 1 / \rho$$

The first partial derivatives

$$Q = e_1 = \left(\frac{\partial e}{\partial q}\right)_{V, s}, \quad -p = e_2 = \left(\frac{\partial e}{\partial V}\right)_{q, s}, \quad T = e_3 = \left(\frac{\partial e}{\partial s}\right)_{q, V}$$

expressed through q, V and s represent equations of state of the medium. They serve as the three missing relationships between thermodynamic quantities.

It is known [1] that at equilibrium state $Q = q' = 0$. Let us select q, V and Q as independent thermodynamic variables. If we assume analytical dependence of q' on Q , then near the equilibrium state we have

$$q' = -H_1(q, V) Q + H_2(q, V) Q^2 + \dots \tag{1.2}$$

From the second law of thermodynamics for irreversible processes it follows that the coefficient $H_1 > 0$.

The third equation of system (1.1) follows from the law of conservation of energy and is transformed into two alternate forms. In the first case we express the increase of specific entropy through the increase in completeness of chemical reaction and the increase in density and pressure

$$ds = \left(\frac{\partial s}{\partial p}\right)_{q, \rho} \left[dp - \left(\frac{\partial p}{\partial q}\right)_{\rho, s} dq - a_f^2 dp \right], \quad a_f = (\partial p / \partial \rho)_{q, s}^{1/2} \tag{1.3}$$

Here a_f is the frozen speed of sound. It follows from (1.3) that

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial r} - a_f^2 \left(\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial r} \right) = \left[\left(\frac{\partial p}{\partial q}\right)_{\rho, s} - \frac{Q}{T} \left(\frac{\partial p}{\partial s}\right)_{q, \rho} \right] \left(\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial r} \right) = L_f$$

Combination of the obtained relationship with equations of continuity and conservation of momentum gives

$$\frac{\partial p}{\partial t} + (v + a_f) \frac{\partial p}{\partial r} + \rho a_f \left[\frac{\partial v}{\partial t} + (v + a_f) \frac{\partial v}{\partial r} + (v - 1) \frac{a_f v}{r} \right] = L_f \tag{1.5}$$

For the derivation of the second alternative form which follows from the law of conservation of energy we take as independent thermodynamic variables the affinity of chemical reaction, the density and pressure. In analogy to the previous case we obtain

$$\frac{\partial p}{\partial t} + (v + a_e) \frac{\partial p}{\partial r} + \rho a_e \left[\frac{\partial v}{\partial t} + (v + a_e) \frac{\partial v}{\partial r} + (v - 1) \frac{a_e v}{r} \right] = \tag{1.6}$$

$$= \left(\frac{\partial p}{\partial Q}\right)_{\rho, s} \left(\frac{\partial Q}{\partial t} + v \frac{\partial Q}{\partial r} \right) - \frac{Q}{T} \left(\frac{\partial p}{\partial s}\right)_{Q, \rho} \left(\frac{\partial q}{\partial t} + v \frac{\partial q}{\partial r} \right) = L_e$$

$$a_e = (\partial p / \partial \rho)_{Q, s}^{1/2}$$

Equations (1.5) and (1.6) are two different but exact forms of notation which express the consequences of the law of conservation of energy. They are analogous to the relationship which is used in the flow theory of inert gases [2] and transform into this relationship for $Q = (\partial p / \partial q)_{\rho, s} = (\partial p / \partial Q)_{\rho, s} = 0$. In this case the operators L_f and L_e become identically zero and both speeds of sound a_f and a_e coincide with the propagation speed of small disturbances.

The following identity is differentiated with respect to ρ

$$p(Q, \rho, s) = p[q(Q, \rho, s), \rho, s]$$

Taking into consideration the definitions for frozen and equilibrium speeds of sound, we have

$$a_f^2 - a_e^2 = -(\partial p / \partial q)_{\rho, s} (\partial q / \partial \rho)_{Q, s} \quad (1.7)$$

$$\left(\frac{\partial q}{\partial \rho}\right)_{Q, s} = \frac{1}{\rho^2} \frac{(\partial Q / \partial V)_{q, s}}{(\partial Q / \partial q)_{V, s}}$$

In accordance with the relationship of Gibbs we have

$$\left(\frac{\partial p}{\partial q}\right)_{\rho, s} = -\left(\frac{\partial^2 e}{\partial q \partial V}\right)_s = -e_{12}, \quad \left(\frac{\partial Q}{\partial V}\right)_{q, s} = \left(\frac{\partial^2 e}{\partial V \partial q}\right)_s = e_{12}$$

$$\left(\frac{\partial Q}{\partial q}\right)_{V, s} = \left(\frac{\partial^2 e}{\partial q^2}\right)_{V, s} = e_{11}$$

Substituting these quantities into Eq. (1.7), we find

$$a_f^2 - a_e^2 = \frac{1}{\rho^2} \frac{e_{12}^2}{e_{11}} \geq 0 \quad (1.8)$$

The inequality sign in (1.8) is determined by the requirement of thermodynamic stability of the system. The equilibrium speed of sound can reach the value of the frozen speed of sound only for the condition $e_{12} = 0$.

2. Asymptotic expansions. Let us now proceed to the examination of limit modes of disturbance propagation, taking advantage of expansion of unknown functions in series with respect to several independent small parameters.

It is known [1, 2] that in relaxing mixtures the transmission of signals is accompanied by dispersion. Moreover, the speed of this transmission in limiting cases coincides either with the frozen, or with the equilibrium speed of sound. The first case is realized when the macroscopic time is much smaller than the time for chemical reaction. The second case corresponds to a process in which the macroscopic time significantly exceeds the time for relaxation. Furthermore, special consideration must be given to media with nearly equal values for both speeds of sound.

Let us assume that the values of all characteristics of the gas mixture at any time and in each point of space deviate little from the corresponding values in the state of rest which represents the state of complete thermodynamic equilibrium. The unperturbed quantities will be denoted by the subscript zero.

We introduce a system of coordinates which moves either with the frozen or the equilibrium speed of sound in the quiescent medium. The characteristic length in this system is denoted by L . We assume that the flow which is examined in the relaxing medium represents a short wave, i.e. the width of the region where the disturbances are concentrated is small in comparison to the disturbances to which the wave propagates. Then the independent variables are

$$t = \frac{L}{\Delta a_0} t', \quad r = a_0 t + L r' \quad (2.1)$$

Here Δ is the small parameter.

With respect to perturbations of the completeness of reaction, the density, pressure and other thermodynamic quantities it is assumed that they are of the same order as the mass velocity of particles which is proportional to the second small parameter ϵ . Passing to dimensionless unknown functions, we have

$$\begin{aligned}
 v &= \varepsilon a_0 v', & q &= q_0 (1 + \varepsilon q'), & \rho &= \rho_0 (1 + \varepsilon \rho'), & p &= p_0 (1 + \varepsilon p') \\
 s &= s_0 (1 + \varepsilon s'), & T &= T_0 (1 + \varepsilon T'), & Q &= \varepsilon \frac{p_0}{q_0 \rho_0} Q' & & \\
 a_f &= a_{f_0} (1 + \varepsilon a_f'), & a_e &= a_{e_0} (1 + \varepsilon a_e') & & & &
 \end{aligned}
 \tag{2.2}$$

As far as the chemical reaction rate is concerned, we have

$$q' = \varepsilon \frac{q_0}{\tau} q'' \tag{2.3}$$

Here τ is the characteristic time for the reaction of components of the mixture.

It is recalled that the speed a_0 of wave motion in the quiescent medium coincides either with the frozen speed a_{f_0} , or the equilibrium speed a_{e_0} of propagation of sound signals.

Substituting Eqs. (2.1) and expansions (2.2) into the initial system of equations (1.1) and taking into account relationships (1.2) and (2.3), an additional numerical parameter appears

$$N_r = L / \tau a_0$$

In the derivation of asymptotic equations in all cases only the principal terms will be retained.

3. Quasi-frozen process. It is assumed that the frozen and the equilibrium speeds of sound in the quiescent medium differ by a finite amount. The short wave which is examined moves, by definition, with a velocity $a_0 = a_{f_0}$ and has a narrow perturbed zone. The prime above dimensionless quantities is subsequently omitted.

After linearization the integration of the first two equations of system (1.1) leads to the following equations

$$\rho = \frac{p_0}{\rho_0 a_0^2} p = v \tag{3.1}$$

The first of these equations expresses the fact that in the approximation which is examined here, the compression of the gas takes place reversibly and with constant composition of the reacting mixture. According to the second equation the Riemann relationship which characterizes a plane running sound impulse in an inert gas [2] applies to the entire flow.

By virtue of smallness of the affinity of chemical reaction we obtain from the third equation of system (1.1)

$$s = 0 \tag{3.2}$$

This supports the conclusion about the reversible character of gas compression.

We write the general expression for the deviation of pressure from its equilibrium value in a quiescent medium, taking as independent thermodynamic parameters the completeness of chemical reaction, the density and entropy

$$p_0 p = (\partial p / \partial q_0)_{\rho, s} q_0 q + \rho_0 a_{f_0}^2 \rho + (\partial p / \partial s_0)_{q, \rho} s_0 s \tag{3.3}$$

Here by virtue of (3.2) the last term in the right side disappears. For the presented relationship to coincide with the first equation (3.1), it is necessary to satisfy the requirement $q = 0$ which was formulated above. This leads to the condition $N_r \ll 1$. In fact, retaining only the principal terms in the fourth equation of system (1.1), we find

$$\partial q / \partial r = -N_r q \quad (3.4)$$

For $N_r \ll 1$ it follows from here that $q = 0$.

The derivation of relationships (3.1) was based on the introduction of the small parameter Δ into the definition of dimensionless time. It follows from the analysis which was performed that in the quasi-frozen mode of disturbance propagation such a substitution is justified only under the condition that the macroscopic time L / a_{f0} is much smaller than the time τ for the chemical reaction.

The presence of the relaxation process is distinctly apparent in the derivation of the last asymptotic equation which makes it possible to establish the dependence of velocity on time and the coordinate. With this purpose we turn to Eq. (1.5). In the approximation which is examined here

$$a_f = \left(\frac{\partial a_f}{\partial \rho_0} \right)_{q,s} \frac{\rho_0}{a_{f0}} \rho = (m_{f0} - 1) \rho, \quad m_{f0} = \frac{1}{2\rho_0^2 a_{f0}^2} \left(\frac{\partial^2 p}{\partial V_0^2} \right)_{q,s}$$

$$Q = \left(\frac{\partial Q}{\partial \rho_0} \right)_{q,s} \frac{q_0 \rho_0^2}{p_0} \rho$$

Let us compare the definitions (1.2) and (2.3) for the rate of chemical reaction. It is clear that

$$H_1 = \frac{q_0^2 \rho_0}{\tau p_0} H_1'$$

Here H_1' is a dimensionless function of the order of unity. Recalling Eq. (1.4) for the operator L_t , we find

$$2\epsilon m_{f0} v \frac{\partial v}{\partial r} + \Delta \left[2 \frac{\partial v}{\partial t} + (v-1) \frac{v}{t} \right] = -N_r \frac{q_0^2 \rho_0 H_{10}}{p_0 a_{f0}^2} \left(\frac{\partial p}{\partial q_0} \right)_{q,s} \left(\frac{\partial Q}{\partial \rho_0} \right)_{q,s} v \quad (3.5)$$

Depending on the relative value of small parameters ϵ , Δ and N_r we can distinguish different cases which are encountered in the investigation of the quasi-frozen mode of propagation of disturbances. Without dwelling on these cases in detail, it is noted that for $N_r \ll \epsilon \sim \Delta$ we can neglect the term in the right side of (3.5). As a result an equation is obtained which defines the flow of an inert gas.

If all terms in Eq. (3.5) are of the same order, then assuming

$$2\epsilon m_{f0} = 2\Delta = N_r \frac{q_0^2 \rho_0 H_{10}}{p_0 a_{f0}^2} \left(\frac{\partial p}{\partial q_0} \right)_{q,s} \left(\frac{\partial Q}{\partial \rho_0} \right)_{q,s}$$

we can reduce Eq. (3.5) to the form

$$v \frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} + \left(1 + \frac{v-1}{2} \frac{1}{t} \right) v = 0 \quad (3.6)$$

The general solution of (3.6) is written in the form

$$r - \tau u = f(u), \quad \tau = \int t^{-(v-1)/2} e^{-t} dt, \quad u = t^{(v-1)/2} e^t v$$

where f is an arbitrary function. The obtained solution allows to formulate basic conclusions of qualitative nature with respect to the quasi-frozen mode of disturbance propagation.

In continuous flows along any characteristic

$$r - vt^{(v-1)/2} e^t \int t^{-(v-1)/2} e^{-t} dt = \text{const} \tag{3.7}$$

the velocity v decays according to the following law

$$v \sim t^{-(v-1)/2} e^{-t}$$

This law was predicted by geometrical acoustics [3, 4], in the framework of which all nonlinear effects are ignored. For $t \rightarrow \infty$ it follows from relationship (3.7) that $r \rightarrow \text{const}$. In other words, all characteristic lines tend to straight lines parallel to the time axis.

Let us examine as an example the plane centered wave which is formed at the instant of time $t = 0$ in the point with the coordinate $r = 0$

$$v|_{t=0} = \begin{cases} 0 & \text{for } r > 1 \\ -2\sigma & \text{for } r < 0 \end{cases} \tag{3.8}$$

In this case $v = 1$ and $\tau = -e^{-t}$. Satisfying the initial condition we find that for $-2\sigma(1 - e^{-t}) \leq r \leq 0$ the function $f = u$. Hence

$$v = r(e^t - 1)^{-1} \tag{3.9}$$

For $r > 0$ the solution is trivial: $v = 0$. For $r < -2\sigma(1 - e^{-t})$ we have $v = -2\sigma e^{-t}$. For small values of t the Eq. (3.9) gives in the first approximation $v = r/t$. As we know [2], this equation describes a centered Riemann wave propagating in an inert gas.

Let the flow now have a discontinuity. In the initial dimensional variables the velocity N for the propagation of a weak shock wave is given by the relationship

$$N = a_{f_0} + \frac{1}{2} \frac{m_{f_0}}{\rho_0 a_{f_0}} (p - p_0)$$

Denoting the coordinate of discontinuity by r_s and the value of the function u immediately behind the discontinuity through u_s , we derive on the basis of Eqs. (2.1), (2.2) and (3.1)

$$dr_s / d\tau = \frac{1}{2} u_s \tag{3.10}$$

In order to obtain the necessary solution of equation (3.6), we assume again $v = 1$ and substitute in the initial condition (3.8) the value -2σ by $+2\sigma$. Then ahead of the shock front $u = 0$ and behind the shock front $u = 2\sigma$. The location of the discontinuity is determined by the solution of Eq. (3.10). As a result

$$r_s = \sigma(1 - e^{-t}) \tag{3.11}$$

In this manner for $t \rightarrow \infty$ the shock wave degenerates into an unperturbed characteristic. Even in the case where in the initial instant behind the discontinuity there is an infinite region with a constant excess pressure, the amplitude of this pressure rapidly becomes zero.

Let us examine the general problem ($v = 1, 2, 3$) of decay of a shock wave, behind the front of which the excess pressure decreases to zero. The change of mixture parameters in the expansion flow depends on the form of function f . The value of u_s for different instants is found from the solution of the ordinary differential equation which follows from (3.10)

$$\left(\frac{df}{du} + \tau \right) \frac{du_s}{d\tau} = -\frac{1}{2} u_s \tag{3.12}$$

In the asymptotic theory of shock wave decay in an inert gas it is established [5, 6] that the distribution of velocity behind the surface of discontinuity changes linearly with respect to the coordinate, i. e. $df/du = \tau_0$. Let us also preserve this law for the relaxing mixture, even though in the case which is under examination the asymptotic behavior of flow parameters for the expansion flow depends to a high degree on initial data. For $df/du = \tau_0$ we have from Eq. (3.12)

$$u_s = c(\tau + \tau_0)^{-1/2}, \quad r_s = c(\tau + \tau_0)^{1/2} \quad c = \text{const} \quad (3.13)$$

For example, at the initial instant $t = 0$ there is a plane wave with the value $v_s = 2\sigma$ for the velocity on the shock front, and with the width λ_0 of the perturbed region. Then the constants $\tau_0 = 1 + \lambda_0(2\sigma)^{-1}$ and $c = \sqrt{2\lambda_0\sigma}$. Utilizing the second equation (3.13), we find that in the limit for $t \rightarrow \infty$ the width of the perturbed region $\lambda_\infty = \lambda_0\sqrt{1 + 2\sigma/\lambda_0}$ increases with respect to the initial width by only a few times. For small values of the ratio $2\sigma/\lambda_0$ the final coordinate of the shock front is $r_s = \sigma$. It is clear that this value is also obtained from Eq. (3.11).

4. Quasi-equilibrium process. Let us assume that the velocity of motion of the short wave which is being examined, in the quiescent medium is equal to the equilibrium speed of sound. After linearization of Euler's continuity equations we obtain again Eqs. (3.1) with the only difference that the quantity a_{f_0} should be replaced by a_{e0} . In other words, in the process under consideration the compression of the gas is accomplished reversibly and for a constant value of affinity of chemical reaction.

As independent thermodynamic variables we select the chemical affinity, the density and specific entropy. The deviation of pressure from the equilibrium value is

$$p_0 p = \left(\frac{\partial p}{\partial Q_0} \right)_{\rho, s} \frac{p_0}{q_0 \rho_0} Q + \rho_0 a_{e0}^2 \rho + \left(\frac{\partial p}{\partial s_0} \right)_{\rho, Q} s_0 s \quad (4.1)$$

The last term in the right side of this relationship disappears by virtue of (3.2). For the change of pressure to take place at a constant value of affinity of chemical reaction, it is now also necessary to convert the first term to zero. This requirement leads to the condition $N_r \gg 1$. In fact, it is evident from (3.4) that for $N_r \gg 1$ it is necessary to set $\dot{q} = 0$. Equation (1.2) shows that together with \dot{q} the chemical affinity $Q = 0$. The obtained results can be summarized in the following manner: in the investigation of the quasi-equilibrium mode of propagation of disturbances in a relaxing medium the introduction of the small parameter Δ in the definition of dimensionless time is permissible under the condition that the macroscopic time L/a_{e0} is significantly greater than the time τ for the chemical reaction.

Let us turn to Eq. (1.6) and derive the missing relationship which is satisfied by the velocity of gas particles. The increase in equilibrium sound velocity is

$$a_e = \left(\frac{\partial a_e}{\partial \rho_0} \right)_{Q, s} \frac{\rho_0}{a_{e0}} \rho = (m_{e0} - 1) \rho, \quad m_{e0} = \frac{1}{2\rho_0^2 a_{e0}^2} \left(\frac{\partial^2 p}{\partial V_0^2} \right)_{Q, s}$$

Therefore the simplification of the left part of Eq. (1.6) is carried out in a manner which is completely analogous to Sect. 3.

For $Q = s = 0$ we have

$$q = \left(\frac{\partial q}{\partial \rho_0} \right)_{Q, s} \frac{\rho_0}{q_0} \rho$$

Utilizing the last relationship, we derive

$$2\varepsilon m_{e0}v \frac{\partial v}{\partial r} + \Delta \left[2 \frac{\partial v}{\partial t} + (v-1) \frac{v}{t} \right] = \quad (4.2)$$

$$= - \frac{1}{N_r} \frac{p_0}{q_0^2 \rho_0 a_{e0}^2 H_{10}} \left(\frac{\partial p}{\partial Q_0} \right)_{\rho, s} \left(\frac{\partial q}{\partial \rho_0} \right)_{Q, s} \frac{\partial^2 v}{\partial r^2}$$

As was shown above, for the quasi-frozen mode the presence of the relaxation process provides an input to the asymptotic equation. This input is proportional to the function v . The change of composition of the reacting mixture in the quasi-equilibrium mode is taken into account with the aid of the second derivative of this function $\partial^2 v / \partial r^2$. The indicated difference has a simple mathematical character. In the first case the small parameter N_r in the initial conditions of Euler stands with unknown functions. In the second case N_r^{-1} plays the role of the small parameter and it is multiplied here by a combination of derivatives of function q .

In problems of nonlinear acoustics, Eq. (4.2) was encountered in the analysis of inert gas flow with viscosity and thermal conductivity. On the basis of this equation, general results were obtained which apply to the decay of plane shock waves in dissipative media [7]. The role of this equation is also noted in the review article [8]. Cylindrically and spherically symmetric shock waves were examined in [9]. Let us introduce into the analysis the Reynolds and Péclet numbers

$$N_{Re1} = \frac{\rho_0 a_0 L}{\lambda_{10}}, \quad N_{Re2} = \frac{\rho_0 a_0 L}{\lambda_{20}}, \quad N_{Pe} = \frac{\rho_0 a_0 c_p L}{k_0}$$

and the dimensionless thermodynamic coefficient

$$m_0 = \frac{1}{2} \rho_0^{-3} a_0^{-2} (\partial^2 p / \partial V_0^2)_s$$

which in the expansion for the speed of sound in an inert gas plays the same role as m_f and m_{e0} . Here λ_1 and λ_2 are the first and second coefficients of viscosity, k is the coefficient of thermal conductivity, c_p is the specific heat at constant pressure, κ is the Poisson isentropic exponent.

The equations which were studied in [7 - 9] agree completely with (4.2) if in the latter a substitution of coefficients is made according to the following rule:

$$m_{e0} \rightarrow m_0, \quad \frac{1}{N_r} \frac{p_0}{q_0^2 \rho_0 a_{e0}^2 H_{10}} \left(\frac{\partial p}{\partial Q_0} \right)_{\rho, s} \left(\frac{\partial q}{\partial \rho_0} \right)_{Q, s} \rightarrow - \frac{1}{N_{Re}} \left(1 + \frac{\kappa_0 - 1}{N_{Pr}} \right) \quad (4.3)$$

Here the total Reynolds number $N_{Re} = (\frac{4}{3} N_{Re1}^{-1} + N_{Re2}^{-1})^{-1}$ is connected with the so-called "longitudinal viscosity". The Prandtl number N_{Pr} is equal to the ratio of Péclet number N_{Pe} to Reynolds number N_{Re} . From this an exact mathematical analogy follows between processes which are being examined. According to this analogy the effect of chemical reactions on quasi-equilibrium propagation of sound impulses is equivalent to the effect of longitudinal viscosity and thermal conductivity on the structure of these impulses.

Another formulation of the analogy between viscous and relaxing flows is widely known. As was pointed out by the authors of [10] and [11] in phenomena of sound propagation the process of chemical reactions formally plays the same role as is fulfilled by the second viscosity which is defined by the coefficient λ_2 . They pointed out the equivalence of corresponding expressions for deviation of pressure from its equilibrium value. These expressions in both cases are proportional to the divergence of velocity. In the investigation of nonlinear motion of short waves it is preferable to formulate the analogy in

terms of the longitudinal viscosity and thermal conductivity. It was established recently [12] that the comparison of stationary transonic flows with flows of inert gas in the transonic range of velocities is also based on a substitution of the type (4.3).

If all terms in Eq. (4.2) are of the same order, then assuming

$$2\epsilon m_{e0} = 2\Delta = -\frac{1}{N_r} \frac{p_0}{q_0^2 \rho_0 a_{e0}^2 H_{10}} \left(\frac{\partial p}{\partial Q_0} \right)_{\rho, s} \left(\frac{\partial q}{\partial \rho_0} \right)_{Q, s}$$

it is possible to reduce this equation to the following form

$$v \frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} + \frac{v-1}{2} \frac{v}{t} = \frac{\partial^2 v}{\partial r^2} \quad (4.4)$$

Let us examine the propagation of plane waves with $v = 1$. In this case the transformation [7, 8]

$$v = -\frac{2}{\Psi} \frac{\partial \Psi}{\partial r}$$

reduces (4.4) to the equation of thermal conductivity.

Let the Cauchy problem (3.8) be formulated for Eq. (4.4). Then we obtain as initial data for the function Ψ

$$\Psi|_{t=0} = \begin{cases} 1 & \text{for } r > 0 \\ e^{\sigma r} & \text{for } r < 0 \end{cases} \quad (4.5)$$

Using the Poisson integral, the solution of the thermal conductivity equation can be written in the form

$$\Psi = \frac{1}{2} \left\{ 1 + \operatorname{erf} \left(\frac{r}{2\sqrt{t}} \right) + e^{\sigma(r+\alpha t)} \left[1 - \operatorname{erf} \left(\sigma\sqrt{t} + \frac{r}{2\sqrt{t}} \right) \right] \right\} \quad (4.6)$$

The computation of asymptotic properties of the solution for $t \rightarrow \infty$ is of greatest interest. It is apparent that for any ray with a positive slope in the rt plane the initial value $\Psi = 1$ does not change in the first approximation; it follows that $v = 0$. On rays with a slope of less than -2σ the velocity v acquires its initial value (3.8), even though the function Ψ differs from (4.5). If

$$r = -2\alpha t, \quad 0 < \alpha < \sigma \quad (4.7)$$

then the evaluation of the principal part of integrals which appear in Eq. (4.6) gives

$$\Psi = \frac{\sigma}{2\sqrt{\pi t} \alpha (\sigma - \alpha)} e^{-\alpha^2 t}$$

It follows from this that for $\alpha \rightarrow 0$ and $\alpha \rightarrow \sigma$ there is a singularity in the solution. Returning to the initial variables we obtain $v = -2\alpha$. The latter equation defines a Riemann wave in the theory of one-dimensional inert gas flow [2]. In fact, this equation shows that along each of straight lines (4.7) the velocity retains a constant value which corresponds to the slope of this straight line.

In this manner, in the case of long times the relaxation process does not have a significant influence on the basic part of the flow. Properties of the perturbed velocity field depend to a great degree on the presence of a chemical reaction only in the vicinity of the front and the tail of the wave the width of which increases with time as \sqrt{t} .

Near the front of the wave we assume

$$r = 2\beta \sqrt{t} \tag{4.8}$$

A simple estimate shows that in the right side of relationship (4.6) we can neglect the second term inside the brackets. As a result we have for the velocity

$$v = - \frac{2}{\sqrt{\pi t} [1 + \operatorname{erf}(\beta)]} e^{-\beta^2} \tag{4.9}$$

According to Eq. (4.9) the decay of disturbances ahead of the wave front takes place in agreement with the exponential law. Behind the wave front for $\beta \rightarrow -\infty$ we have $v = 2\beta / \sqrt{t}$, i.e. the solution which is being examined gives the distribution of gas parameters in the Riemann wave with a constant value of velocity on rays in the rt -plane.

The character of behavior of disturbances in the vicinity of the tail of the wave is established in an analogous manner. Here

$$r = -2\sigma t + 2\beta \sqrt{t}$$

In this region in the first approximation we have

$$v = -2\sigma + \frac{2}{\sqrt{\pi t} [1 - \operatorname{erf}(\beta)]} e^{-\beta^2} \tag{4.10}$$

The relationship (4.10) shows that behind the shock front the velocity approaches exponentially its limiting value -2σ . In the region ahead of the front for $\beta \rightarrow +\infty$ we find $v = -2\sigma + 2\beta / \sqrt{t}$, i.e. the change of velocity obeys the laws for simple waves.

Let us compare the formulated solution of equation (4.4) with the solution which was obtained within the framework of acoustics [13]. The fundamental difference here is that in the linear theory the fan of straight lines with a constant value of velocity along each of these lines is absent. At the same time the influence of the relaxation process which produces a diffusion of the boundaries of the perturbed region according to Eqs. (4.9) and (4.10), is taken into account correctly by this theory. From this it is clear that acoustics predict the growth of the width of the perturbed zone as \sqrt{t} .

In the conditions of Cauchy (3.8) let us now replace the value -2σ by $+2\sigma$. In order to utilize previous results, it is sufficient to introduce the minus sign in front of the constant σ in Eqs. (4.5) and (4.6). Let us elucidate the limit properties of the solution for $t \rightarrow \infty$. In the rt -plane along any ray for which the slope is greater than 2σ , the initial value $\Psi = 1$ is preserved in the first approximation, i.e. $v = 0$. On all rays with a negative slope the velocity v is equal to its initial value (3.8) regardless of differences in the initial and current values of the function Ψ . If we write $r = 2\alpha t$ ($0 < \alpha < \sigma$), then, retaining in Eq. (4.6) only the principal terms, we have $\Psi = 1 + e^{-\sigma(r-\alpha t)}$ for any α . It follows from this that

$$v = \sigma - \sigma \operatorname{th} \frac{\sigma(r - \alpha t)}{2} \tag{4.11}$$

This solution describes the structure of a weak shock wave moving with the velocity σ . It was first found by Taylor [14] in the study of flow of a viscous and heat conducting gas.

Let us examine the region in which

$$r = 2\sigma t + 2\beta \sqrt{t} \tag{4.12}$$

Here in the first approximation

$$v = \sigma [1 - \operatorname{erf}(\beta)] e^{-\sigma^2 t - 2\sigma\beta \sqrt{t}}$$

For large positive values of β we obtain

$$v = \frac{\sigma}{\beta \sqrt{\pi}} e^{-(\sigma \sqrt{t} + \beta)^2} \tag{4.13}$$

Let the coordinate r now be given by Eq. (4.8). Then

$$v = 2\sigma - \sigma [1 + \operatorname{erf}(\beta)] e^{-\sigma^2 t + 2\sigma\beta \sqrt{t}}$$

For large negative values of β we derive

$$v = 2\sigma + \frac{\sigma}{\beta \sqrt{\pi}} e^{-(\sigma \sqrt{t} - \beta)^2} \tag{4.14}$$

Equations (4.13) and (4.14) show that the limit values of velocity ahead and behind the shock wave are reached very rapidly, because the exponent contains in addition to β also the time t . It is easy to verify that the solutions for regions where the coordinate is given by equations (4.8) or (4.12) transform into Taylor's solution (4.11). In the first case it is necessary to take the parameter β to the limit $-\infty$, in the second case to $+\infty$.

5. Media with nearly equal speeds of sound. Let us recall equation (1.8) which determines the difference between the squares of frozen and equilibrium speeds of sound. If their values in the medium at rest are nearly equal, then the thermodynamic derivative

$$e_{120} = \epsilon_a \frac{p_0}{q_0} e'_{120}$$

where ϵ_a is a new small parameter, e'_{120} is a dimensionless quantity of the order of unity. The necessity for investigation of media of this type was first noted in [15].

From the last equation of system (1.1), which gives the rate for the relaxation process, it is evident that the perturbed completeness of reaction is proportional to the product $\epsilon \epsilon_a$. The functions Q and q' must also be proportional to this product. Therefore, in the transition to dimensionless variables and the subsequent simplification of equations of motion for the mixture, it is necessary in relationships (2.2) and (2.3) to make the substitution

$$q' \rightarrow \epsilon_a q', \quad Q' \rightarrow \epsilon_a Q', \quad q'' \rightarrow \epsilon_a q''$$

As a result we have

$$Q = \frac{q_0^2 \epsilon_a e_{110}}{p_0} q - e_{120} Q \tag{5.1}$$

As for all dimensionless quantities, the prime above e_{120} was left out here. Both values for the speed of sound a_{f0} and a_{e0} are nearly equal, therefore we can drop the assumption that the velocity α_0 of wave motion necessarily coincides with one of them. We assume according to relationship (1.8)

$$a_0 - a_{f_0} = \epsilon_a^2 a_0 \sigma_{f_0}, \quad a_0 - a_{e_0} = \epsilon_a^2 a_0 \sigma_{e_0} \tag{5.2}$$

The constants σ_{f_0} and σ_{e_0} are apparently of the order of unity.

The first two equations of the initial system (1.1) do not contain functions q and Q . The linearization of these functions leads again to Eq. (3.1) with the substitution of a_{f_0} by a_0 . It is evident that Eq. (3.2) for the entropy increase is also valid in the case of special media which is being investigated. A more accurate evaluation shows that $s \sim \epsilon \epsilon_a^2$.

Let us consider expansions (3.3) and (4.1) which give the deviation of pressure from the equilibrium value in the quiescent medium. The thermodynamic derivative has the form

$$\left(\frac{\partial p}{\partial Q}\right)_{\rho, s} = \frac{(\partial p / \partial q)_{V, s}}{(\partial Q / \partial q)_{V, s}}$$

Therefore, with accuracy to quantities of the order of ϵ_a^2 both expansions coincide with the first of Eqs. (3.1). In this case the dimensionless parameter N_r remains arbitrary.

Equation (3.4) for the rate of chemical reaction is presented in the form

$$\frac{\partial q}{\partial r} = N_r H_{10} \left(\frac{q_0^2 \rho_0 \epsilon_{110}}{p_0} q - e_{120} v \right) \tag{5.3}$$

For the derivation of the missing relationship we take advantage again of alternate forms of (1.5) and (1.8) which follow from the law of conservation of energy. If the difference between the velocity a_0 of wave motion and the frozen sound speed a_{f_0} is taken into account, then the first of the equations gives

$$2(\epsilon m_{f_0} v - \epsilon_a^2 \sigma_{f_0}) \frac{\partial v}{\partial r} + \Delta \left[2 \frac{\partial v}{\partial t} + (v - 1) \frac{v}{t} \right] = \epsilon_a^2 \frac{p_0 \epsilon_{120}}{\rho_0 a_0^3} \frac{\partial q}{\partial r} \tag{5.4}$$

In a completely analogous manner, taking into account the difference between a_0 and a_{e_0} , we derive from the second alternate form

$$-2(\epsilon m_{e_0} v - \epsilon_a^2 \sigma_{e_0}) \frac{\partial v}{\partial r} + \Delta \left[\frac{\partial v}{\partial t} + (v - 1) \frac{v}{t} \right] = \epsilon_a^2 \frac{p_0^2 \epsilon_{120}}{q_0^2 \rho_0^2 \epsilon_{110} a_0^2} \frac{\partial Q}{\partial r} \tag{5.5}$$

Equations (5.4) and (5.5) must coincide with each other.

To become convinced of this, we first of all note the equality $m_{f_0} = m_{e_0} = m_0$ which is satisfied with accuracy to ϵ_a^2 . Recalling further the definition (5.2) of constants σ_{f_0} and σ_{e_0} , we find

$$2(\sigma_{f_0} - \sigma_{e_0}) = - \frac{p_0^2 \epsilon_{120}^2}{q_0^2 \rho_0^2 \epsilon_{110} a_0^2}$$

Substituting now expressions (5.1) for Q into the right side of equation (5.5), we bring it to the form (5.4). Eliminating from equations (5.3) and (5.4) the thermodynamic function q , we arrive at an equation of the second order which contains only the perturbed particle velocity of the mixture

$$2(\epsilon m_{f_0} v - \epsilon_a^2 \sigma_{e_0}) \frac{\partial v}{\partial r} + \Delta \left[2 \frac{\partial v}{\partial t} + (v - 1) \frac{v}{t} \right] - \epsilon_a^2 \frac{1}{N_r} \frac{p_0^3 \epsilon_{120}^2}{q_0^4 \rho_0^3 \epsilon_{110}^2 H_{10} a_0^2} \frac{\partial^2 v}{\partial r^2} = \\ = \frac{1}{N_r} \frac{p_0}{q_0^2 \rho_0^2 \epsilon_{110} H_{10}} \frac{\partial}{\partial r} \left\{ 2(\epsilon m_{f_0} v - \epsilon_a^2 \sigma_{e_0}) \frac{\partial v}{\partial r} + \Delta \left[2 \frac{\partial v}{\partial t} + (v - 1) \frac{v}{t} \right] \right\} \tag{5.6}$$

In the limiting cases Eq. (5.6) transforms into (3.5) or (4.2).

Let $a_0 = a_{f0}$, $N_r \rightarrow 0$ and $\varepsilon_a^2 N_r \sim \varepsilon$. Discarding minor terms in Eq. (5.6) and integrating the resulting relationship, we obtain Eq. (3.5) for $a_{f0} \rightarrow a_{e0}$.

Let us examine the other possibility when $a_0 = a_{e0}$, $N_r^{-1} \rightarrow 0$ and $\varepsilon_a^2 N_r^{-1} \sim \varepsilon$. In this case we obtain Eqs. (4.2), in the right side of which only the principal term is retained for the condition $a_{f0} \rightarrow a_{e0}$.

The minor parameters are subjected to the following relationships:

$$\varepsilon m_0 = \Delta = \varepsilon_a^2 \frac{p_0^2 \varepsilon_{120}^2}{q_0^2 \rho_0^2 \varepsilon_{110} a_0^2}$$

We introduce the notation

$$\sigma = \frac{\varepsilon_a^2}{\varepsilon} \frac{\sigma_{e0}}{m_0}, \quad l = \frac{1}{N_r} \frac{p_0}{q_0^2 \rho_0 \varepsilon_{110} H_{10}}$$

Changing to the new unknown function $v - \sigma = u$, we write Eq. (5.6) as

$$u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} + \frac{v-1}{2} \frac{u+\sigma}{t} = l \frac{\partial}{\partial r} \left(u \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} + \frac{v-1}{2} \frac{u+\sigma}{t} \right) + \frac{l}{2} \frac{\partial^2 u}{\partial r^2}$$

The obtained equation is applied to the description of internal structure of shock waves. For this purpose let us examine its solution for $v = 1$, which is independent of time. For completely dispersed waves without discontinuities we have after a single integration

$$lu \frac{du}{dr} + \frac{1}{2} l \frac{du}{dr} - \frac{1}{2} u^2 = -\frac{1}{2} \sigma^2 \quad (5.7)$$

The arbitrary constant is determined here from the condition that $u \rightarrow -\sigma$ for $r \rightarrow \infty$. If the wave is not completely dispersed, the region of perturbations is bounded by the front. In the transition through the front the completeness of reaction is preserved and the velocity experiences a jump. Nevertheless, relationship (5.7) retains its validity for discontinuous motions of a relaxing medium.

In order to become convinced of this, we write the following equation in the initial variables

$$j^2 = -(p - p_0) / (V - V_0) \quad (5.8)$$

This equation gives the mass flux j in a system of coordinates which moves with the front of the wave. Equation (5.8) is well known in the theory of shock waves [2], but it can also be applied in the investigation of continuous flows. Let us expand the pressure in a series in which we select as independent variables the completeness of chemical reaction, the specific volume and entropy. We retain terms not only of the first order, but also of the second order of smallness. After transition to dimensionless quantities we find

$$p_0 p = -\varepsilon_a^2 p_0 \varepsilon_{120} q - p_0 a_{f_0}^2 V + \frac{1}{2} \varepsilon \frac{1}{\rho_0^2} \left(\frac{\partial^2 p}{\partial V_0^2} \right)_{q, s} V^2$$

If we take into account that the flux $j = -\rho_0 a_0$, then substitution of the expansion into the right side of (5.8) gives the following relationship between the functions q and v in any point of the perturbed region:

$$\varepsilon_a^2 \sigma_{f_0} = \frac{1}{2} \varepsilon m_0 \left(v - \frac{q_0^2 \rho_0 \varepsilon_{110}}{p_0 \varepsilon_{120}} \frac{q}{v} \right)$$

It is essential to note that the presence of this relationship does not depend on whether both functions which are being examined are continuous or contain discontinuities which correspond to a shock front with a coordinate $r = r_s$. Substituting v by u , we have

$$\frac{q_0^2 \rho_0 \epsilon_{110}}{\rho_0 \epsilon_{120}} q = u^2 + u - \sigma^2 + \sigma \quad (5.9)$$

As a result of substitution of this equation into the right side of Eq. (5.3) we obtain relationship (5.7), i.e. the statement formulated above is proven. This indicates that expression (5.9) is an exact integral of Eq. (5.3).

On the shock front the completeness of reaction does not change. It follows from this that

$$u_s = \sigma - 1 \quad (5.10)$$

We can obtain the last equation in a different way. From the equation which gives the velocity N for the propagation of a weak shock wave we derive

$$dr_s / dt = 1/2 (u_s + 1 - \sigma)$$

The solution which is being examined does not depend on time, i.e. $dr_s / dt = 0$. This leads to (5.10).

Integration of equation (5.7) gives the following relationship

$$\frac{r - r_s}{l} = \ln (\sigma + u)^{1-1/2\sigma} (\sigma - u)^{1+1/2\sigma}$$

This equation is applicable to the description of both completely dispersed shock waves and also waves with incomplete dispersion.

The problem of shock wave structure is treated somewhat differently in [13].

Many of the results presented in this paper were obtained independently by V. V. Lunev to whom the author expresses his sincere appreciation for extended discussions.

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Translated by B. D.

ON A CERTAIN NONLINEAR EQUATION OF ELECTROHYDRODYNAMICS AND MAGNETOELASTICITY

PMM Vol. 35, №6, 1971, pp. 1038 - 1046

K. Sh. KHODZHAEV

(Leningrad)

(Received January 1, 1971)

Properties of the solutions of the Emden-Fowler equation the nonlinear term of which contains the unknown function raised to a negative power, are determined. Boundary value problems in which one of the conditions corresponds to the requirement that the solution be bounded when the argument is equal to zero or infinity (this requirement occurs in a number of problems in mechanics) are also considered. These boundary value problems may have any number, or even an enumerable set of solutions, the latter case characterized by the dependence of these solutions on a parameter of an unusual form.

For $n > 0$ (n is the power in which the unknown function appears in the nonlinear term) the Emden-Fowler equation has been studied exhaustively in [1, 2]. The problems of electrohydrodynamics and nonlinear magnetoelasticity leading to the Emden-Fowler equation with $n = -2$, were studied in [3 - 5]. The present paper deals with yet another problem, namely that of equilibrium of heavy filaments through which a current flows. This problem leads to the case of $n = -1$ and the nonlinear term may be positive or negative, depending on whether the filaments attract or repel each other.

1. Reduction to an autonomous system and properties of its solutions. The Emden-Fowler equation has the form [1]

$$\frac{d}{d\rho} \left(\rho^\alpha \frac{dw}{d\rho} \right) \pm \rho^\alpha w^n = 0 \quad (1.1)$$